Functions

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C O N N E X I O N S

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Collection structure revised: February 1, 2011
PDF generated: February 6, 2011
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Chapter 1

Function Concepts – Introduction[1]

The unit on functions is the most important in the Algebra II course, because it provides a crucial transition point. Roughly speaking…

- Before Algebra I, math is about **numbers**.
- Starting in Algebra I, and continuing into Algebra II, math is about **variables**.
- Beginning with Algebra II, and continuing into Calculus, math is about **functions**.

Each step builds on the previous step. Each step expands the ability of mathematics to model behavior and solve problems. And, perhaps most crucially, each step can be frightening to a student. It can be very intimidating for a beginning Algebra student to see an entire page of mathematics that is covered with letters, with almost no numbers to be found!

Unfortunately, many students end up with a very vague idea of what variables are ("That's when you use letters in math") and an even more vague understanding of functions ("Those things that look like $f(x)$ or something"). If you leave yourself with this kind of vague understanding of the core concepts, the lessons will make less and less sense as you go on: you will be left with the feeling that "I just can't do this stuff" without realizing that the problem was all the way back in the idea of a variable or function.

The good news is, variables and functions both have very specific meanings that are not difficult to understand.

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[1] This content is available online at <http://cnx.org/content/m18192/1.4/>.

Chapter 2

Function Concepts – What is a Variable?[1]

A variable is a letter that stands for a number you don't know, or a number that can change.

A few examples:

Example 2.1: Good Examples of Variable Definitions

- "Let p be the number of people in a classroom."
- "Let A be John's age, measured in years."
- "Let h be the number of hours that Susan has been working."

In each case, the letter stands for a very specific number. However, we use a letter instead of a number because we don't know the specific number. In the first example above, different classrooms will have different numbers of people (so p can be different numbers in different classes); in the second example, John's age is a specific and well-defined number, but we don't know what it is (at least not yet); and in the third example, h will actually change its value every hour. In all three cases, we have a good reason for using a letter: it represents a number, but we cannot use a specific number such as "–3" or " $4\frac{1}{2}$".

Example 2.2: Bad Examples of Variable Definitions

- "Let n be the nickels."
- "Let M be the number of minutes in an hour."

The first error is by far the most common. Remember that a variable always stands for a number. "The nickels" are not a number. Better definitions would be: "Let n be the number of nickels" or "Let n be the total value of the nickels, measured in cents" or "Let n be the total mass of the nickels, measured in grams."

The second example is better, because "number of minutes in an hour" is a number. But there is no reason to call it "The Mysterious Mr. M" because we already know what it is. Why use a letter when you just mean "60"?

Bad variable definitions are one of the most common reasons that students get stuck on word problems—or get the wrong answer. The first type of error illustrated above leads to variable confusion: n will end up being used for "number of nickels" in one equation and "total value of the nickels" in another, and you end up with the wrong answer. The second type of error is more harmless—it won't lead to wrong answers—but it won't help either. It usually indicates that the student is asking the wrong question ("What can I assign a variable to?") instead of the right question ("What numbers do I need to know?")

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[1] This content is available online at <http://cnx.org/content/m18194/1.5/>.

2.1 Variables aren't all called x. Get used to it.

Many students expect all variables to be named x, with possibly an occasional guest appearance by y. In fact, variables can be named with practically any letter. Uppercase letters, lowercase letters, and even Greek letters are commonly used for variable names. Hence, a problem might start with "Let H be the home team's score and V be the visiting team's score."

If you attempt to call both of these variables x, it just won't work. You could in principle call one of them x and the other y, but that would make it more difficult to remember which variable goes with which team. It is important to become comfortable using a wide range of letters. (I do, however, recommend avoiding the letter o whenever possible, since it looks like the number 0.)

Chapter 3

Function Concepts – What is a Function?[1]

A function is neither a number nor a variable: it is a **process for turning one number into another**. For instance, "Double and then add 6" is a function. If you put a 4 into that function, it comes out with a 14. If you put a $\frac{1}{2}$ into that function, it comes out with a 7.

The traditional image of a function is a machine, with a slot on one side where numbers go in and a slot on the other side where numbers come out.

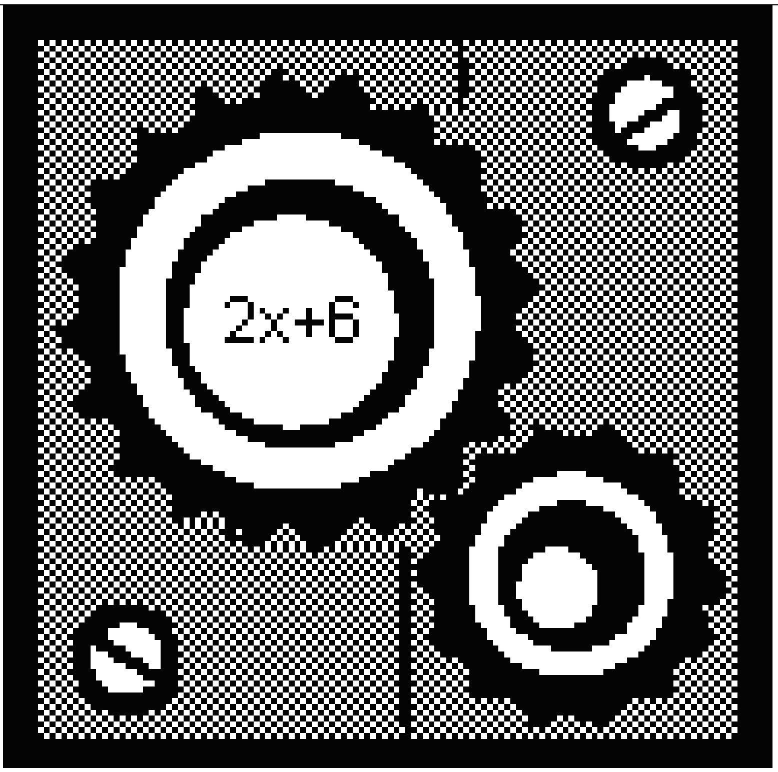
5 →		→ 16

Table 3.1: A number goes in. A number comes out. The function is the machine, the process that turns 4 into 14 or 5 into 16 or 100 into 206.

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[1] This content is available online at <http://cnx.org/content/m18189/1.2/>.

The point of this image is that the function is not the numbers, but the machine itself—the **process**, not the results of the process.

The primary purpose of "The Function Game" that you play on Day 1 is to get across this idea of a numerical process. In this game, one student (the "leader") is placed in the role of a function. "Whenever someone gives you a number, you double that number, add 6, and give back the result." It should be very clear, as you perform this role, that you are **not** modeling a number, a variable, or even a list of numbers. You are instead modeling a process—or an algorithm, or a recipe—for turning numbers into other numbers. That is what a function is.

The function game also contains some more esoteric functions: "Respond with –3 no matter what number you are given," or "Give back the lowest prime number that is greater than or equal to the number you were given." Students playing the function game often ask "Can a function **do** that?" The answer is always yes (with one caveat mentioned below). So another purpose of the function game is to expand your idea of what a function can do. Any process that consistently turns numbers into other numbers, is a function.

By the way—having defined the word "function" I just want to say something about the word "equation." An "equation" is when you "equate" two things—that is to say, set them equal. So $x^2 - 3$ is a function, but it is not an equation. $x^2 - 3 = 6$ is an equation. An "equation" always has an equal sign in it.

Chapter 4

Function Concepts – The Rule of Consistency[1]

There is only one limitation on what a function can do: **a function must be consistent.**

For instance, the function in the above drawing is given a 5, and gives back a 16. That means this particular function turns 5 into 16—always. That particular function can never take in a 5 and give back a 14. This "rule of consistency" is a very important constraint on the nature of functions.

> NOTE: **This rule does not treat the inputs and outputs the same!**

For instance, consider the function $y = x^2$. This function takes both 3 and -3 and turns them into 9 (two different inputs, same output). That is allowed. However, it is not reversible! If you take a 9 and turn it into both a 3 and a –3 (two different outputs, same input), you are not a function.

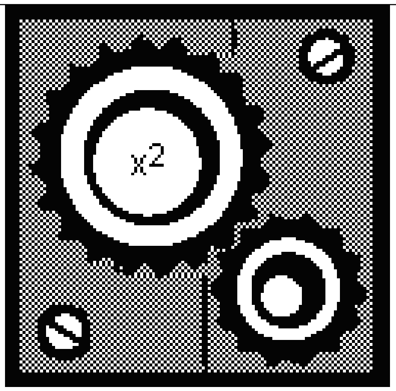
$3 \rightarrow$ $-3 \rightarrow$		$\rightarrow 9$

Table 4.1: If 3 goes in, 9 comes out. If –3 goes in, 9 also comes out. No problem: x^2 is a function.

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[1] This content is available online at <http://cnx.org/content/m18190/1.3/>.

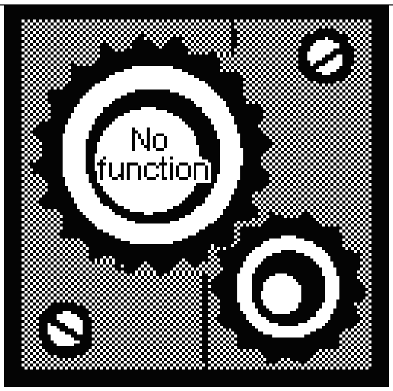
→ 9		3 → −3 →

Table 4.2: If 9 goes in, both −3 and 3 come out. This **violates the rule of consistency:** no function can do this

This asymmetry has the potential to cause a great deal of confusion, but it is a very important aspect of functions.

Chapter 5

Function Concepts – Four Ways to Represent a Function[1]

Modern Calculus texts emphasize that a function can be expressed in four different ways.

1. **Verbal** - This is the first way functions are presented in the function game: "Double and add six."
2. **Algebraic** - This is the most common, most concise, and most powerful representation: $2x + 6$. Note that in an algebraic representation, the input number is represented as a variable (in this case, an x).
3. **Numerical** - This can be done as a list of value pairs, as $(4, 14)$ — meaning that if a 4 goes in, a 14 comes out. (You may recognize this as (x, y) points used in graphing.)
4. **Graphical** - This is discussed in detail in the section on graphing.

These are **not** four different types of functions: they are four different views of the same function. One of the most important skills in Algebra is converting a function between these different forms, and this theme will recur in different forms throughout the text.

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[1] This content is available online at <http://cnx.org/content/m18195/1.3/>.

Chapter 6

Function Concepts – Domain and Range[1]

Consider the function $y = \sqrt{x}$. If this function is given a 9 it hands back a 3. If this function is given a 2 it hands back…well, it hands back $\sqrt{2}$, which is approximately 1.4. The answer cannot be specified exactly as a fraction or decimal, but it is a perfectly good answer nonetheless.

On the other hand, what if this function is handed –4? There is no $\sqrt{-4}$, so the function has no number to hand back. If our function is a computer or calculator, it responds with an error message. So we see that this function is able to respond to the numbers 9 and 2, but it is not able to respond in any way to the number -4. Mathematically, we express this by saying that 9 and 2 are in the "domain" of the square root function, and –4 is **not** in the domain of this function.

Definition 6.1: Domain

The **domain** of a function is all the numbers that it can successfully act on. Put another way, it is all the numbers that can go into the function.

A square root cannot successfully act on a negative number. We say that "The domain of $\sqrt{x}$ is all numbers Image not finished such that Image not finished" meaning that if you give this function zero or a positive number, it can act on it; if you give this function a negative number, it cannot.

A subtler example is the function $y = \sqrt{x+7}$. Does this function have the same domain as the previous function? No, it does not. If you hand this function a –4 it successfully hands back $\sqrt{3}$ (about 1.7). –4 is in the domain of this function. On the other hand, if you hand this function a –8 it attempts to take $\sqrt{-1}$ and fails; –8 is **not** in the domain of this function. If you play with a few more numbers, you should be able to convince yourself that the domain of this function is all numbers x such that $x \geq -7$.

You are probably familiar with two mathematical operations that are **not allowed**. The first is, you are not allowed to take the square root of a negative number. As we have seen, this leads to restrictions on the domain of any function that includes square roots.

The second restriction is, you are not allowed to divide by zero. This can also restrict the domain of functions. For instance, the function $y = \frac{1}{x^2-4}$ has as its domain all numbers **except** $x = 2$ and $x = -2$. These two numbers both cause the function to attempt to divide by 0, and hence fail. If you ask a calculator to plug $x = 2$ into this function, you will get an error message.

So: if you are given a function, how can you find its domain? Look for any number that puts a negative number under the square root; these numbers are **not** in the domain. Look for any number that causes the function to divide by zero; these numbers are not in the domain. All other numbers **are** in the domain.

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[1] This content is available online at <http://cnx.org/content/m18191/1.4/>.

Function	**Domain**	**Comments**
$\sqrt{x}$	$x \geq 0$	You can take the square root of 0, or of any positive number, but you cannot take the square root of a negative number.
$\sqrt{x+7}$	$x \geq -7$	If you plug in any number **greater than or equal to –7,** you will be taking a legal square root. If you plug in a number **less than –7**, you will be taking the square root of a negative number.This domain can also be understood graphically: the graph $y = \sqrt{x}$ has been moved 7 units to the left. See "horizontal permutations" below.
$\frac{1}{x}$	$x \neq 0$	In other words, the domain is "all numbers except 0." You are not allowed to divide by 0. You are allowed to divide by anything else.
$\frac{1}{x-3}$	$x \neq 3$	If $x = 3$ then you are dividing by 0, which is **not** allowed. If $x = 0$you are dividing by –3, which is allowed. So be careful! The rule is **not** "when you are dividing, x cannot be 0." The rule is " x can never be any value that would put a 0 in the denominator."

continued on next page

$\frac{1}{x^2-4}$	$x \neq \pm 2$	Or, " $x$can be any number except 2 or –2." Either of these x values will put a 0 in the denominator, so neither one is allowed.
$2^x + x^2 - 3x + 4$	All numbers	You can plug any x value into this function and it will come back with a number.
$\frac{\sqrt{x-3}}{x-5}$	x $\geq$ 3 x $\neq$ 5	In words, the domain is **all numbers greater than or equal to 3, except the number 5**. Numbers less than 3 put negative numbers under the square root; 5 causes a division by 0.

Table 6.1

You can confirm all these results with your calculator; try plugging numbers into these functions, and see when you get errors!

A related concept is **range**.

Definition 6.2: Range

The **range** of a function is all the numbers that it may possibly produce. Put another way, it is all the numbers that can come **out** of the function.

To illustrate this example, let us return to the function $y = \sqrt{x+7}$. Recall that we said the domain of this function was all numbers x such that $x \geq -7$; in other words, you are allowed to put any number greater than or equal to –7 into this function.

What numbers might come out of this function? If you put in a –7 you get out a 0. ($\sqrt{0} = 0$) If you put in a –6 you get out $\sqrt{1} = 1$. As you increase the x value, the y values also increase. However, if you put in $x = -8$ nothing comes out at all. Hence, the range of this function is all numbers y such that $y \geq 0$. That is, this function is capable of handing back 0 or any positive number, but it will never hand back a negative number.

It's easy to get the words domain and range confused—and it's important to keep them distinct, because although they are **related** concepts, they are **different** from each other. One trick that sometimes helps is to remember that, in everyday useage, "your domain" is your home, your land—it is where you begin. A function begins in its own domain. It ends up somewhere out on the range.

6.1 A different notation for domain and range

Domains and ranges above are sometimes expressed as intervals, using the following rules:

- Parentheses () mean "an interval starting or ending here, but not including this number"
- Square brackets [] mean "an interval starting or ending here, including this number"

This is easiest to explain with examples.

This notation...	**...means this...**	**...or in other words**
$(-3, 5)$	All numbers between –3 and 5, **not including** –3 and 5.	$-3 < x < 5$
$[-3, 5]$	All numbers between –3 and 5, **including** –3 and 5.	$-3 \leq x \leq 5$
$[-3, 5)$	All numbers between –3 and 5, **including** –3 **but not** 5.	$-3 \leq x < 5$
$(-\infty, 10]$	All numbers less than or equal to 10.	$x \leq 10$
$(23, \infty)$	All numbers greater than 23.	$x > 23$
$(-\infty, 4)$ $(4, \infty)$	All numbers less than 4, and all numbers greater than 4. In other words, **all numbers except** 4.	$x \neq 4$

Table 6.2

Chapter 7

Function Concepts – Functions in the Real World[1]

Why are functions so important that they form the heart of math from Algebra II onward?

Functions are used whenever **one variable depends on another variable.** This relationship between two variables is the most important in mathematics. It is a way of saying “If you tell me what x is, I can tell you what y is.” We say that y “depends on” x, or y “is a function of” x.

A few examples:

Example 7.1: Function Concepts – Functions in the Real World

- "The area of a circle depends on its radius."
- "The amount of money Alice makes depends on the number of hours she works."
- “Max threw a ball. The height of the ball depends on how many seconds it has been in the air.”

In each case, there are two variables. Given enough information about the scenario, you could assert that **if you tell me this variable, I will tell you that one**. For instance, suppose you know that Alice makes $100 per day. Then we could make a chart like this.

If Alice works this many days...	**...she makes this many dollars**
0	0
1	100
$1\frac{1}{2}$	150
8	800

Table 7.1

If you tell me how long she has worked, I will tell you how much money she has made. Her earnings “depend on” how long she works.

The two variables are referred to as the **dependent variable** and the **independent variable**. The dependent variable is said to “depend on” or “be a function of” the independent variable. “The height of the ball is a function of the time.”

Example 7.2: Bad Examples of Functional Relationships

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[1] This content is available online at <http://cnx.org/content/m18193/1.2/>.

- "The number of Trojan soldiers depends on the number of Greek soldiers."
- "The time depends on the height of the ball."

The first of these two examples is by far the most common. It is simply not true. There may be a relationship between these two quantities—for instance, the **sum** of these two variables might be the total number of soldiers, and the **difference** between these two quantities might suggest whether the battle will be a fair one. But there is no **dependency** relationship—that is, no way to say “If you tell me the number of Greek soldiers, I will tell you the number of Trojan soldiers”—so this is not a function.

The second example is subtler: it confuses the **dependent** and the **independent** variables. The height depends on the time, not the other way around. More on this in the discussion of “Inverse Functions".

Chapter 8

Function Concepts – Function Notation[1]

8.1 Function Notation

Functions are represented in math by parentheses. When you write $f(x)$ you indicate that the variable f is a function of—or depends on—the variable x.

For instance, suppose $f(x) = x^2 + 3x$. This means that f is a function that takes whatever you give it, and squares it, and multiplies it by 3, and adds those two quantities.

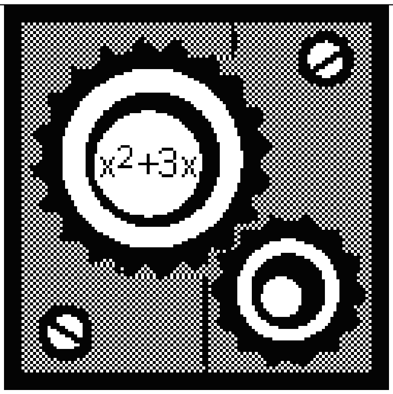
$7 \rightarrow$ $10 \rightarrow$ $x \rightarrow$ $y \rightarrow$ a dog $\rightarrow$		$\rightarrow f(7) = 7^2 + 3(7) = 70$ $\rightarrow f(10) = 10^2 + 3(10) = 130$ $\rightarrow f(x) = x^2 + 3x$ $\rightarrow f(y) = y^2 + 3y$ $\rightarrow f(\text{dog}) = (\text{dog})^2 + 3(\text{dog})$ (*not in the domain)

Table 8.1

The notation $f(7)$ means "plug the number 7 into the function f." It does **not** indicate that you are multiplying f times 7. To evaluate $f(7)$ you take the function $f(x)$ and replace all occurrences of the variable x with the number 7. If this function is given a 7 it will come out with a 70.

If we write $f(y) = y^2 + 3y$ we have **not specified a different function**. Remember, the function is not the variables or the numbers, it is the process. $f(y) = y^2 + 3y$ also means "whatever number comes in, square it, multiply it by 3, and add those two quantities." So it is a different way of writing the same function.

Just as many students expect all variables to be named x, many students—and an unfortunate number of parents—expect all functions to be named f. The correct rule is that—whenever possible—**functions, like variables, should be named descriptively**. For instance, if Alice makes \$100/day, we might write:

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[1] This content is available online at <http://cnx.org/content/m18188/1.8/>.

- Let m equal the amount of money Alice has made (measured in dollars)
- Let t equal the amount of time Alice has worked (measured in days)
- Then, $m(t) = 100t$

This last equation should be read " m is a function of t (or m depends on t). Given any value of the variable t, you can multiply it by 100 to find the corresponding value of the variable m."

Of course, this is a very simple function! While simple examples are helpful to illustrate the concept, it is important to realize that very complicated functions are also used to model real world relationships. For instance, in Einstein's Special Theory of Relativity, if an object is going very fast, its mass is multiplied by $\frac{1}{\sqrt{1-\frac{v^2}{9\cdot 10^{16}}}}$. While this can look extremely intimidating, it is just another function. The speed v is the independent variable, and the mass m is dependent. Given any speed v you can determine how much the mass m is multiplied by.

Chapter 9

Function Concepts – Algebraic Generalizations[1]

When you have a "generalization," you have one **broad fact** that allows you to assume many **specific** facts as examples.

Example 9.1
Generalization: "Things fall down when you drop them."

Specific facts, or examples:

- Leaves fall down when you drop them
- Bricks fall down when you drop them
- Tennis balls fall down when you drop them

If **any one** of the individual statements does not work, the generalization is invalid. (This generalization became problematic with the invention of the helium balloon.)

Scientists tend to work **empirically,** meaning they start with the specific facts and work their way back to the generalization. Generalizations are valued in science because they bring order to apparently disconnected facts, and that order in turn suggests underlying theories.

Mathematicians also spend a great deal of time looking for generalizations. When you have an "algebraic generalization" you have one **algebraic** fact that allows you to assume many **numerical** facts as examples.

Consider, for instance, the first two functions in the function game.

1. Double the number, then add six.
2. Add three to the number, then double.

These are very different "recipes." However, their inclusion in the function game is a bit unfair, because—here comes the generalization—**these two functions will always give the same answer.** Whether the input is positive or negative, integer or fraction, small or large, these two functions will mimic each other perfectly. We can express this generalization in words.

Example 9.2
Generalization: If you plug a number into the function **double and add six**, and plug the same number into the function **add three and double**, the two operations will give the same answer.

Specific facts, or examples:

- If you double –5 and add six; or, if you add –5 to 3 and then double; you end up with the same answer.

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[1] This content is available online at <http://cnx.org/content/m18186/1.4/>.

- If you double 13 and add six; or, if you add 13 to 3 and then double; you end up with the same answer.

There is literally an infinite number of specific claims that fit this pattern. We don't need to prove or test each of these claims individually: once we have proven the generalization, we know that **all** these facts must be true.

We can express this same generalization pictorially by showing two "function machines" that always do the same thing.

$-5 \rightarrow$		$\rightarrow 2\,(-5)+6=-4$
$0 \rightarrow$		$\rightarrow 2\,(0)+6=6$
$13 \rightarrow$		$\rightarrow 2\,(13)+6=32$

Table 9.1

$-5 \rightarrow$		$\rightarrow 2\,(-5)+6=-4$
$0 \rightarrow$		$\rightarrow 2\,(0)+6=6$
$13 \rightarrow$		$\rightarrow 2\,(13)+6=32$

Table 9.2

But the most common way to express this generalization is algebraically, by asserting that these two functions **equal** each other.

$$2x+6=2\,(x+3) \qquad (9.1)$$

Many beginning Algebra II students will recognize this as the distributive property. Given $2\left(x+3\right)$ they can correctly turn it into $2x+6$. **But they often fail to realize what this equality means—that given the same input, the two functions will always yield the same output.**

Example 9.3
Generalization: $2x+6=2\left(x+3\right)$

Specific facts, or examples:

- $\left(2\times -5\right)+6=2\times\left(-5+3\right)$
- $\left(2\times 0\right)+6=2\times\left(0+3\right)$
- $\left(2\times 13\right)+6=2\times\left(13+3\right)$

It's worth stopping for a moment here to think about the = symbol. Whenever it is used, = indicates that two things are the same. However, the following two equations use the = in very different ways.

$$2x^2+5x=3 \tag{9.2}$$

$$\frac{2x^2-18}{x+3}=2x-6 \tag{9.3}$$

In the first equation, the = challenges you to **solve for** x. "Find all the x values that make this equation true." The answers in this case are $x=\frac{1}{2}$ and $x=-3$. If you plug in either of these two x-values, you get a true equation; for any other x-value, you get a false equation.

The second equation cannot be solved for x; the = sign in this case is asserting an equality that is true for **any** x-value. Let's try a few.

Example 9.4
Generalization: $\frac{2x^2-18}{x+3}=2x-6$
Specific facts, or examples:

$x=3$	$\frac{2(3)^2-18}{(3)+3}=\frac{18-18}{6}=0$	$2\left(3\right)-6=0$ ✓
$x=-2$	$\frac{2(-2)^2-18}{(-2)+3}=\frac{8-18}{1}=-10$	$2\left(-2\right)-6=-10$ ✓
$x=0$	$\frac{2(0)^2-18}{(0)+3}=\frac{0-18}{3}=-6$	$2\left(0\right)-6=-6$ ✓
$x=\frac{1}{2}$	$\frac{2\left(\frac{1}{2}\right)^2-18}{\left(\frac{1}{2}\right)+3}=\frac{\frac{1}{2}-18}{\frac{7}{2}}=\left(\frac{-35}{2}\right)\left(\frac{2}{7}\right)=-5$	$2\left(\frac{1}{2}\right)-6=-5$ ✓

Table 9.3

With a calculator, you can attempt more difficult values such as $x=-26$ or $x=\pi$; in every case, the two formulas will give the same answer. When we assert that two very different functions will always produce the same answers, we are making a very powerful generalization.

Exception: $x=-3$ is outside the domain of one of these two functions. In this important sense, the two functions are not in fact equal. Take a moment to make sure you understand why this is true!

Such generalizations are very important because they allow us to **simplify**.

Suppose that you were told "I am going to give you a hundred numbers. For each number I give you, square it, then double the answer, then subtract eighteen, then divide by the original number plus three." This kind of operation comes up all the time. But you would be quite relieved to discover that you can accomplish the same task by simply doubling each number and subtracting 6! The generalization in this case is $\frac{2x^2-18}{x+3}=2x-6$; you will be creating exactly this sort of generalization in the chapter on Rational Expressions.

Chapter 10

Function Concepts – Graphing[1]

Graphing, like algebraic generalizations, is a difficult topic because many students know **how to do it** but are not sure **what it means**.

For instance, consider the following graph:

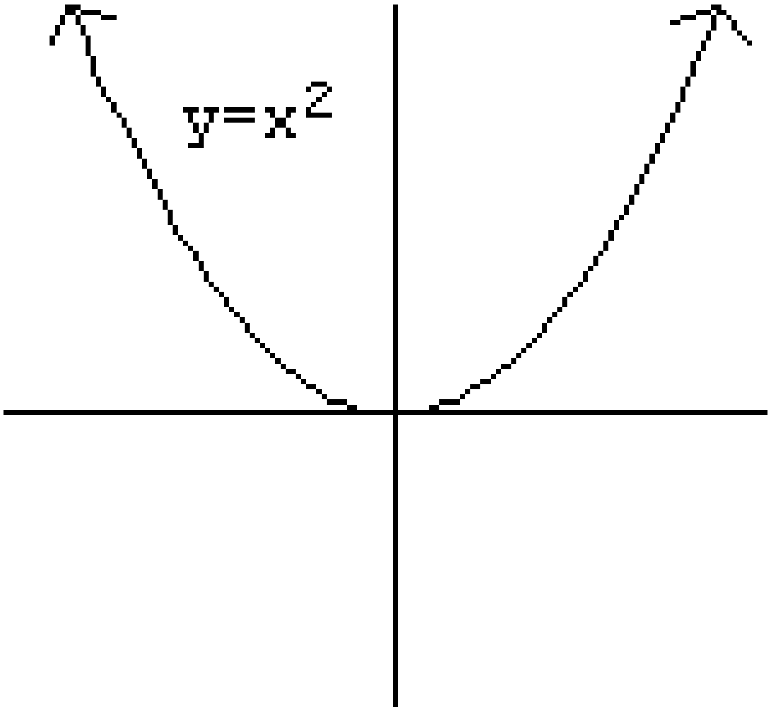

Figure 10.1

If I asked you "Draw the graph of $y = x^2$" you would probably remember how to plot points and draw the shape.

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[1] This content is available online at <http://cnx.org/content/m18196/1.4/>.

But suppose I asked you this instead: "Here's a function, $y = x^2$. And here's a shape, that sort of looks like a U. What do they actually have to do with each other?" This is a harder question! What does it **mean** to graph a function?

The answer is simple, but it has important implications for a proper understanding of functions. Recall that every point on the plane is designated by a unique (x, y) pair of coordinates: for instance, one point is $(5, 3)$. We say that its x -value is 5 and its y -value is 3.

A few of these points have the particular property that their y -values are the square of their x -values. For instance, the points $(0, 0)$, $(3, 9)$, and $(-5, 25)$ all have that property. $(5, 3)$ and $(-2, -4)$ do not.

The graph shown—the pseudo-U shape—is **all the points in the plane that have this property**. Any point whose y-value is the square of its x-value is on this shape; any point whose y-value is not the square of its x-value is not on this shape. Hence, glancing at this shape gives us a complete visual picture of the function $y = x^2$ **if we know how to interpret it correctly**.

10.1 Graphing Functions

Remember that every function specifies a relationship between two variables. When we graph a function, we put the **independent variable** on the x-axis, and the **dependent variable** on the y-axis.

For instance, recall the function that describes Alice's money as a function of her hours worked. Since Alice makes \$12/hour, her financial function is $m(t) = 12t$. We can graph it like this.

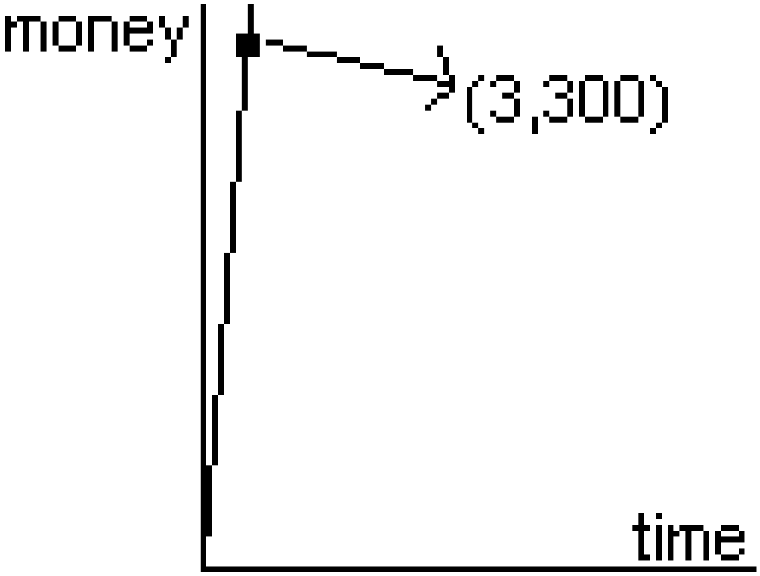


Figure 10.2

This simple graph has a great deal to tell us about Alice's job, if we read it correctly.

- The graph contains the point $(3, 300)$.What does that tell us? That after Alice has worked for three hours, she has made \$300.
- The graph goes through the origin (the point $(0, 0)$). What does that tell us? That when she works 0 hours, Alice makes no money.

- The graph exists only in the first quadrant. What does that tell us? On the mathematical level, it indicates the **domain** of the function ($t \geq 0$) and the **range** of the function ($m \geq 0$). In terms of the situation, it tells us that Alice cannot work negative hours or make negative money.
- The graph is a straight line. What does that tell us? That Alice makes the same amount of money every day: every day, her money goes up by \$100. (\$100/day is the **slope** of the line—more on this in the section on linear functions.)

Consider now the following, more complicated graph, which represents **Alice's hair length as a function of time** (where time is now measured in weeks instead of hours).

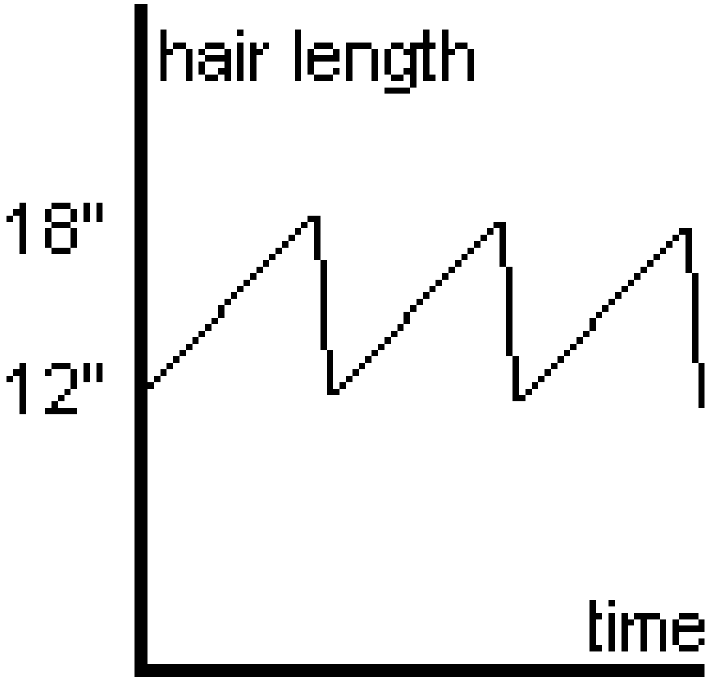


Figure 10.3

What does this graph $h(t)$ tell us? We can start with the same sort of simple analysis.

- The graph goes through the point $(0, 12)$.This tells us that at time ($t = 0$), Alice's hair is 12" long.
- The range of this graph appears to be $12 \leq h \leq 18$. Alice never allows her hair to be shorter than 12" or longer than 18".

But what about the shape of the graph? The graph shows a gradual incline up to 18", and then a precipitous drop back down to 12"; and this pattern repeats throughout the shown time. The most likely explanation is that Alice's hair grows slowly until it reaches 18", at which point she goes to the hair stylist and has it cut down, within a very short time (an hour or so), to 12". Then the gradual growth begins again.

10.2 The rule of consistency, graphically

Consider the following graph.

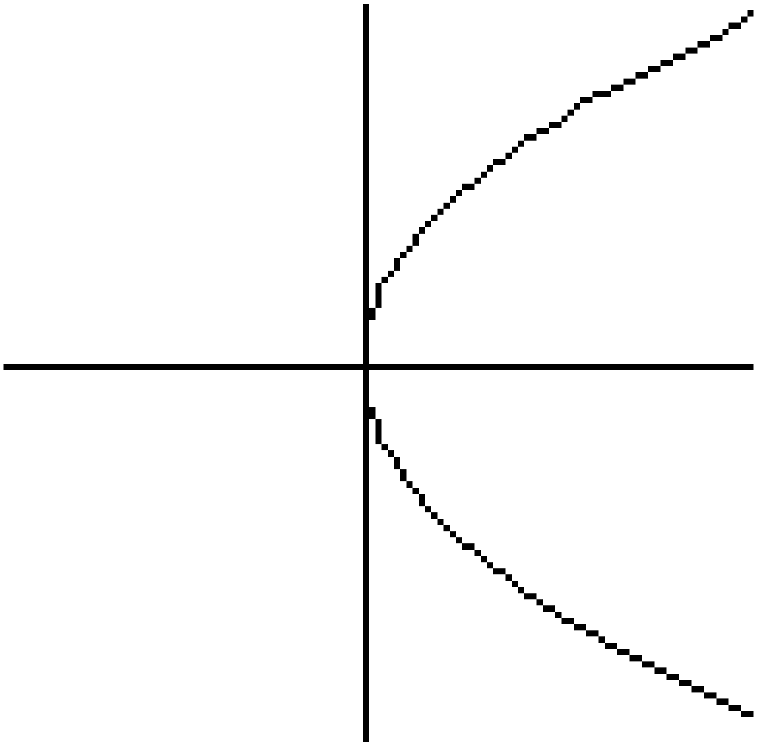


Figure 10.4

This is our earlier "U" shaped graph ($y = x^2$) turned on its side. This might seem like a small change. But ask this question: what is y when $x = 3$? This question has two answers. This graph contains the points $(3, -9)$ and $(3, 9)$. So when $x = 3$, y is both 9 and –9 on this graph.

This violates the only restriction on functions—the rule of consistency. Remember that the x-axis is the independent variable, the y-axis the dependent. In this case, one "input" value (3) is leading to two different "output" values $(-9, 9)$ We can therefore conclude that this graph **does not represent a function at all.** No function, no matter how simple or complicated, could produce this graph.

This idea leads us to the "vertical line test," the graphical analog of the rule of consistency.

Definition 10.1: The Vertical Line Test

If you can draw **any** vertical line that touches a graph in two places, then that graph violates the rule of consistency and therefore does not represent any function.

It is important to understand that the vertical line test is not a new rule! It is the graphical version of the rule of consistency. If any vertical line touches a graph in two places, then the graph has two different y-values for the same x-value, and this is the only thing that functions are not allowed to do.

10.3 What happens to the graph, when you add 2 to a function?

Suppose the following is the graph of the function $y = f(x)$.

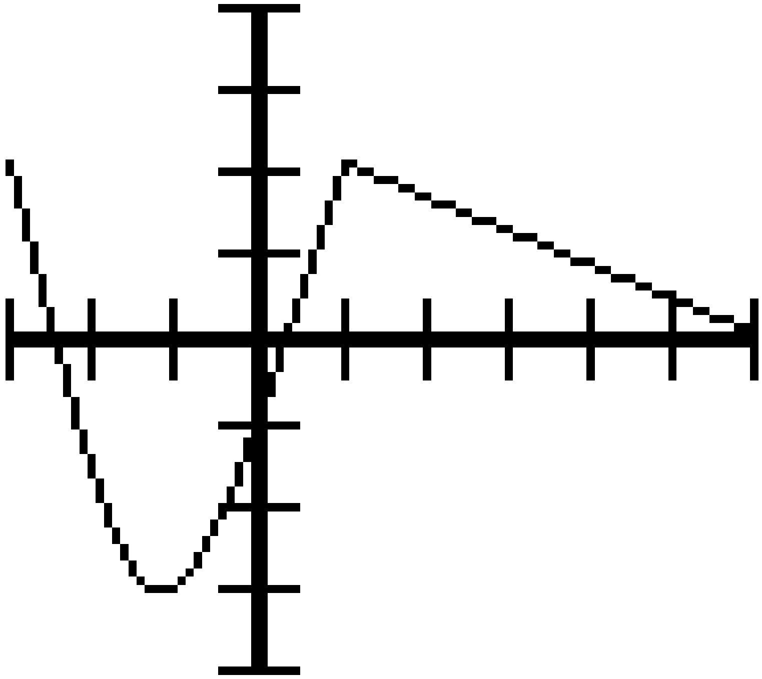


Figure 10.5: $y = f(x)$; Contains the following points (among others): $(-3,2)$, $(-1,-3)$, $(1,2)$, $(6,0)$

We can see from the graph that the domain of the graph is $-3 \leq x \leq 6$ and the range is $-3 \leq y \leq 2$.

Question: **What does the graph of** $y = f(x) + 2$ **look like**?

This might seem an impossible question, since we do not even know what the function $f(x)$ is. But we don't need to know that in order to plot a few points.

x	$f(x)$	$f(x+2)$	**so** $y = f(x)$ **contains this point**	**and** $y = f(x) + 2$ **contains this point**
–3	2	4	$(-3,2)$	$(-3,4)$
–1	–3	–1	$(-1,-3)$	$(-1,-1)$
1	2	4	$(1,2)$	$(1,4)$
6	0	2	$(6,0)$	$(6,2)$

Table 10.1

If you plot these points on a graph, the pattern should become clear. Each point on the graph is **moving up by two**. This comes as no surprise: since you added 2 to each y-value, and adding 2 to a y-value moves any point up by 2. So the new graph will look identical to the old, only moved up by 2.

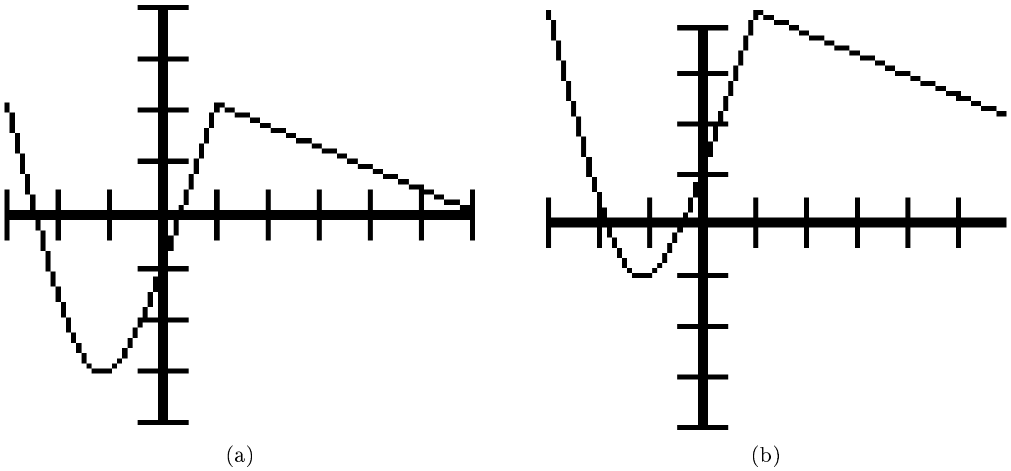
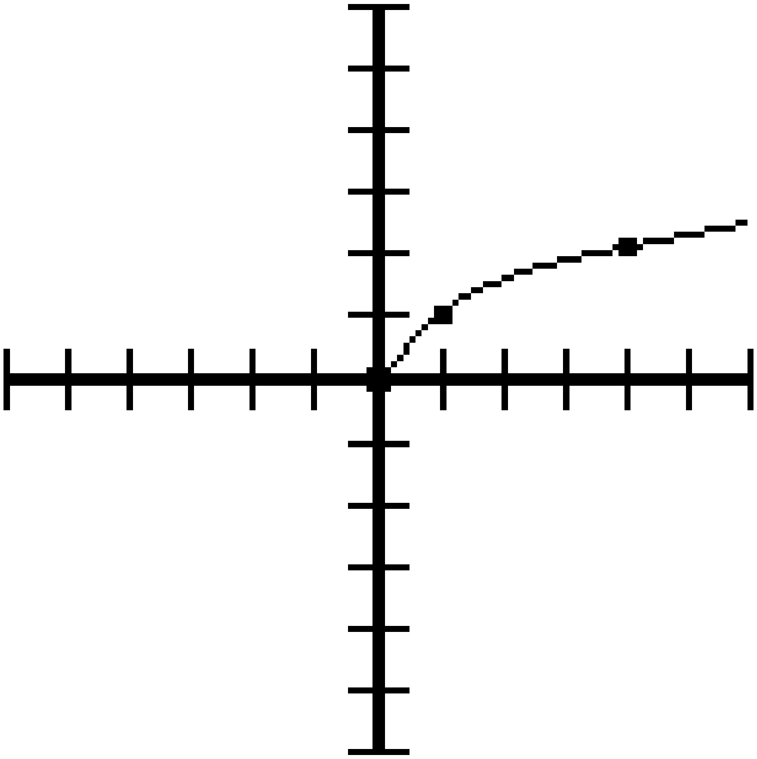

(a) (b)

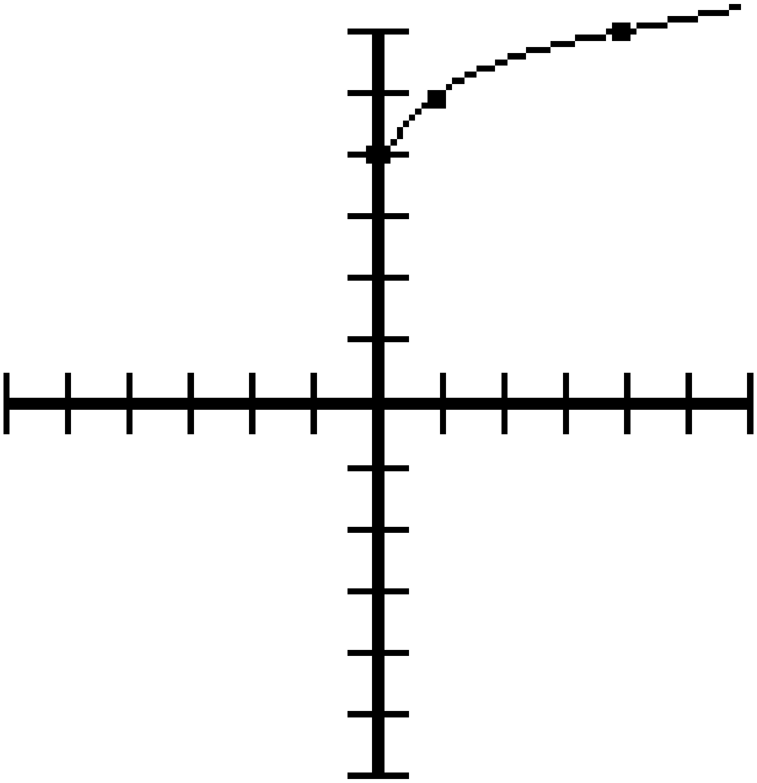
Figure 10.6: (a) $y = f(x)$ (b) $y = f(x) + 2$; All y-values are 2 higher

In a similar way, it should be obvious that if you **subtract** 10 from a function, the graph moves **down** by 10. Note that, in either case, the **domain** of the function is the same, but the **range** has changed.

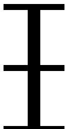
These permutations work for **any function**. Hence, given the graph of the function $y = \sqrt{x}$ below (which you could generate by plotting points), you can produce the other two graphs **without** plotting points, simply by moving the first graph up and down.



(a)



(b)



10.4 Other vertical permutations

Adding or subtracting a constant from $f(x)$, as described above, is one example of a **vertical permutation:** it moves the graph up and down. There are other examples of vertical permutations.

For instance, what does **doubling** a function do to a graph? Let's return to our original function:

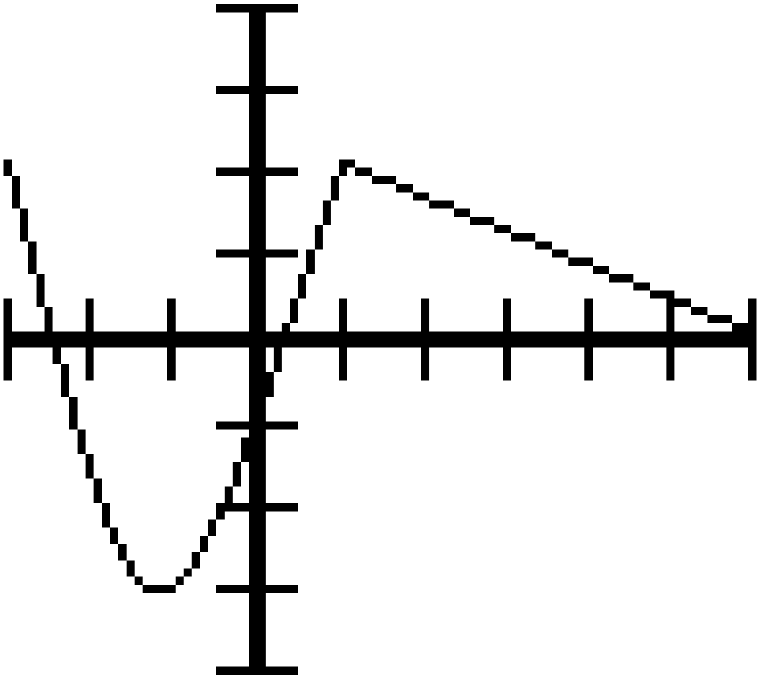


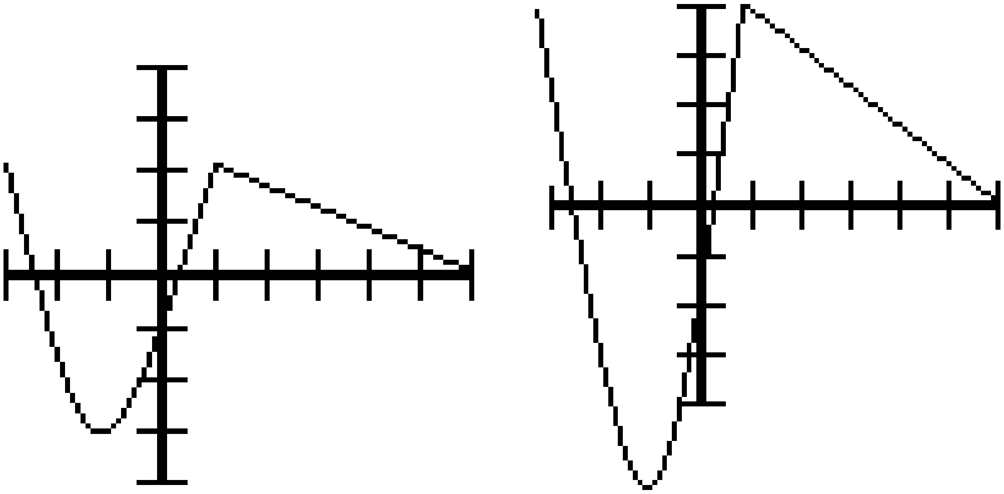
Figure 10.8: $y = f(x)$

What does the graph $y = 2f(x)$ look like? We can make a table similar to the one we made before.

x	$f(x)$	$2f(x)$	**so** $y = 2f(x)$ **contains this point**
–3	2	4	$(-3, 4)$
–1	–3	–6	$(-1, -6)$
1	2	4	$(1, 4)$
6	0	0	$(6, 0)$

Table 10.2

In general, the high points move higher; the low points move lower. The entire graph is **vertically stretched**, with each point moving farther away from the x-axis.

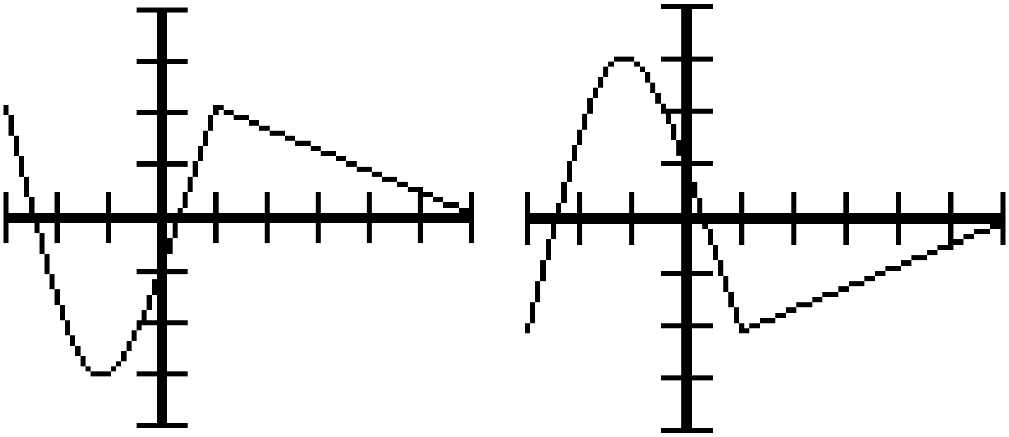


(a) (b)

Figure 10.9: (a) $y = f(x)$ (b) $y = 2f(x)$; All y-values are doubled

Similarly, $y = \frac{1}{2}f(x)$ yields a graph that is vertically compressed, with each point moving toward the x-axis.

Finally, what does $y = -f(x)$ look like? All the positive values become negative, and the negative values become positive. So, point by point, the entire graph flips over the x-axis.



(a) (b)

Figure 10.10: (a) $y = f(x)$ (b) $y = -f(x)$; All y-values change sign

10.5 What happens to the graph, when you add 2 to the x value?

Vertical permutations affect the y-value; that is, the output, or the function itself. Horizontal permutations affect the x-value; that is, the numbers that come in. They often do the opposite of what it naturally seems they should.

Let's return to our original function $y = f(x)$.

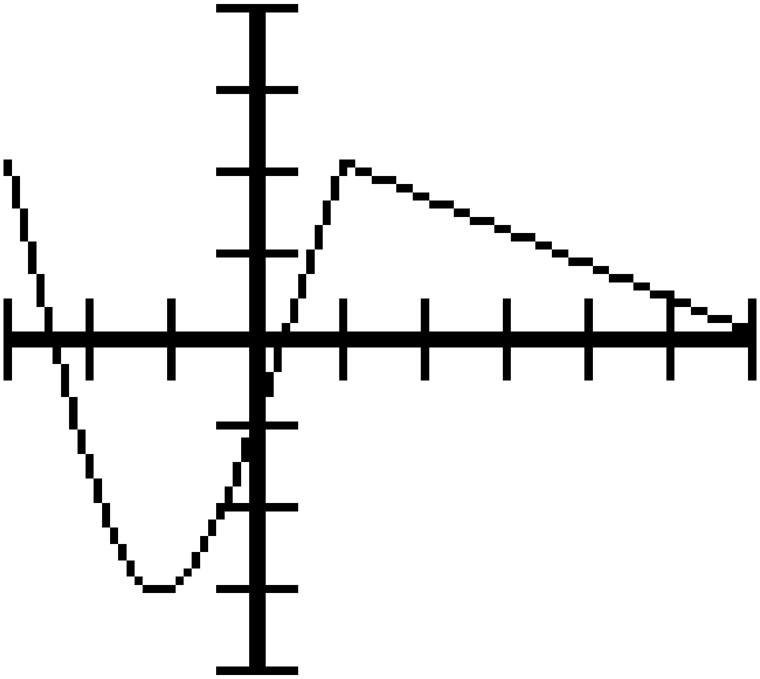


Figure 10.11: $y = f(x)$; Contains the following points (among others): $(-3, 2)$, $(-1, -3)$, $(1, 2)$, $(6, 0)$

Suppose you were asked to graph $y = f(x+2)$. Note that this is **not the same** as $f(x) + 2$! The latter is an instruction to run the function, and then add 2 to all results. But $y = f(x+2)$ is an instruction to add 2 to every x-value **before plugging it into the function.**

- $f(x) + 2$ changes y, and therefore shifts the graph vertically
- $f(x+2)$ changes x, and therefore shifts the graph horizontally.

But which way? In analogy to the vertical permutations, you might expect that adding two would shift the graph to the right. But let's make a table of values again.

x	$x+2$	$f(x+2)$	**so $y = f(x+2)$ contains this point**
–5	–3	f(–3)=2	$(-5, 2)$
–3	–1	f(–1)=–3	$(-3, -3)$
–1	1	f(1)=2	$(-1, 2)$
4	6	f(6)=0	$(4, 0)$

Table 10.3

This is a very subtle, very important point—please follow it closely and carefully! First of all, make sure you understand where all the numbers in that table came from. Then look what happened to the original graph.

> NOTE: **The original graph** $f(x)$ **contains the point** $(6,0)$; **therefore,** $f(x+2)$ **contains the point** $(4,0)$. The point has moved two spaces to the left.

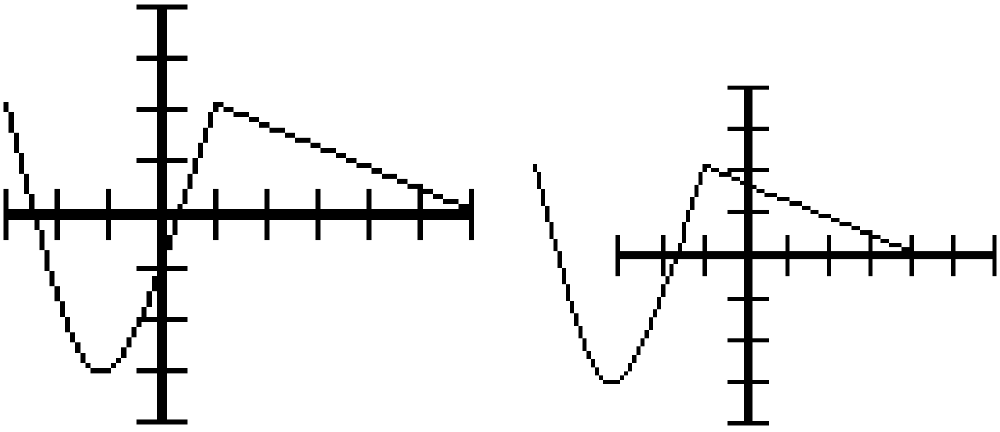

(a) (b)

Figure 10.12: (a) $y = f(x)$ (b) $y = f(\mathrm{x}+2)$; Each point is shifted to the left

You see what I mean when I say horizontal permutations "often do the opposite of what it naturally seems they should"? **Adding** two moves the graph to the **left**.

Why does it work that way? Here is my favorite way of thinking about it. $f(x-2)$ is an instruction that says to each point, "look two spaces to your left, and copy what the original function is doing **there**." At $x=5$ it does what $f(x)$ does at $x=3$. At $x=10$, it copies $f(8)$. And so on. Because it is always copying $f(x)$ to its **left**, this graph ends up being a copy of $f(x)$ moved to the **right**. If you understand this way of looking at it, all the rest of the horizontal permutations will make sense.

Of course, as you might expect, subtraction has the opposite effect: $f(x-6)$ takes the original graph and moves it 6 units to the **right**. In either case, these horizontal permutations affect the **domain** of the original function, but not its **range**.

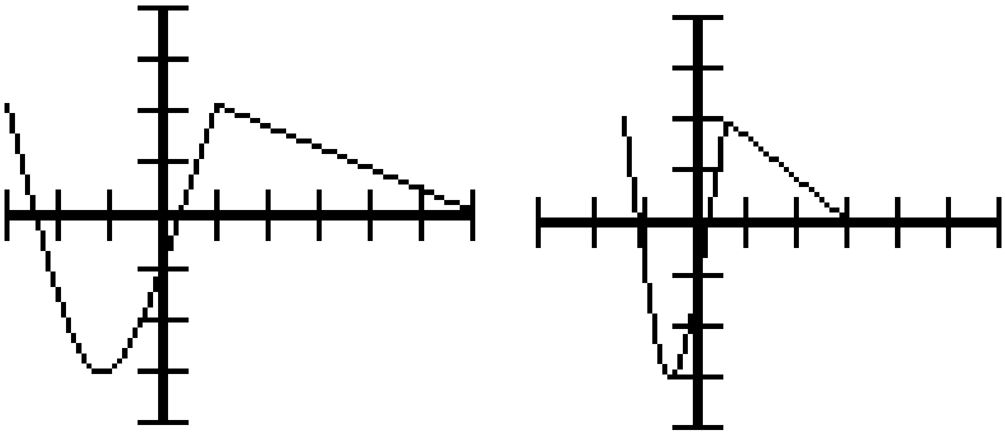
10.6 Other horizontal permutations

Recall that $y = 2f(x)$ **vertically stretches** a graph; $y = \frac{1}{2}f(x)$ **vertically compresses**. Just as with addition and subtraction, we will find that the horizontal equivalents work backward.

x	$2x$	$f(2x)$	**so $y = 2f(x)$ contains this point**
$-1\frac{1}{2}$	-3	2	$\left(-1\frac{1}{2}, 2\right)$
$-\frac{1}{2}$	-1	-3	$\left(-\frac{1}{2}; -3\right)$
$\frac{1}{2}$	1	2	$\left(\frac{1}{2}; 2\right)$
3	6	0	$(3, 0)$

Table 10.4

The original graph $f(x)$ contains the point $(6, 0)$; therefore, $f(2x)$ contains the point $(3, 0)$. Similarly, $(-1; -3)$ becomes $\left(-\frac{1}{2}; -3\right)$. Each point is closer to the y-axis; the graph has horizontally **compressed**.

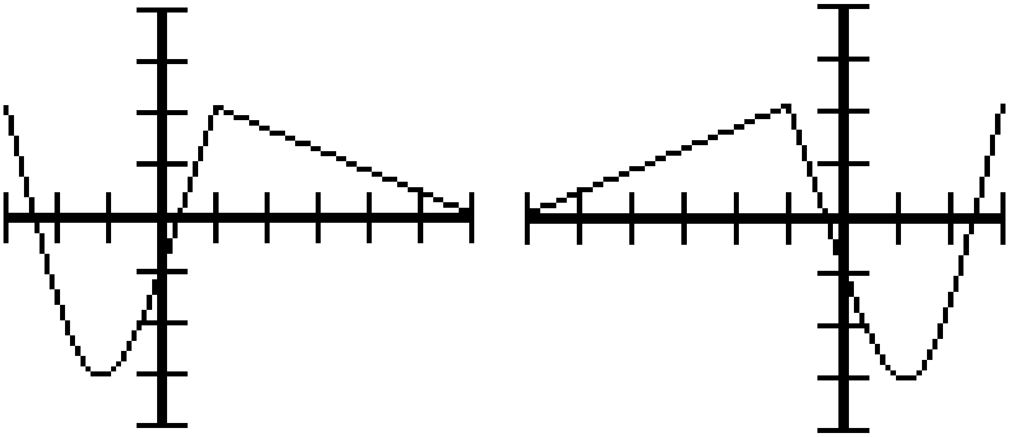


(a) (b)

Figure 10.13: (a) $y = f(x)$ (b) $y = f(2\text{x})$; Each point is twice as close to the y-axis

We can explain this the same way we explained $f(x-2)$. In this case, $f(2x)$ is an instruction that says to each point, "Look outward, at the x-value that is double yours, and copy what the original function is doing **there**." At $x = 5$ it does what $f(x)$ does at $x = 10$. At $x = -3$, it copies $f(-6)$. And so on. Because it is always copying f(x) **outside** itself, this graph ends up being a copy of $f(x)$ moved **inward**; ie a compression. Similarly, $f\left(\frac{1}{2}x\right)$ causes each point to look inward toward the y-axis, so it winds up being a **horizontally stretched** version of the original.

Finally, $y = f(-x)$ does precisely what you would expect: it flips the graph around the y-axis. $f(-2)$ is the old $f(2)$ and vice-versa.

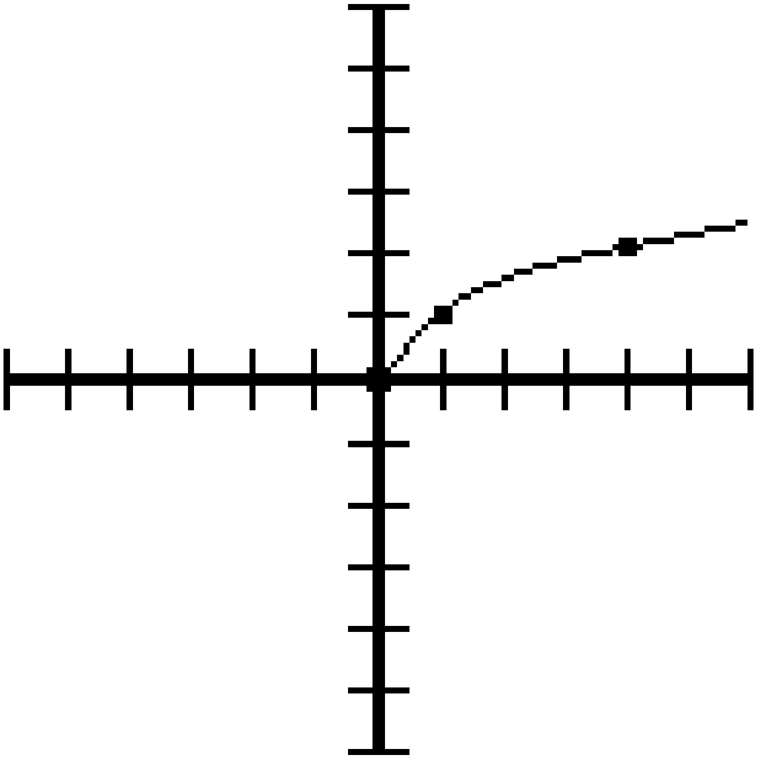


(a) (b)

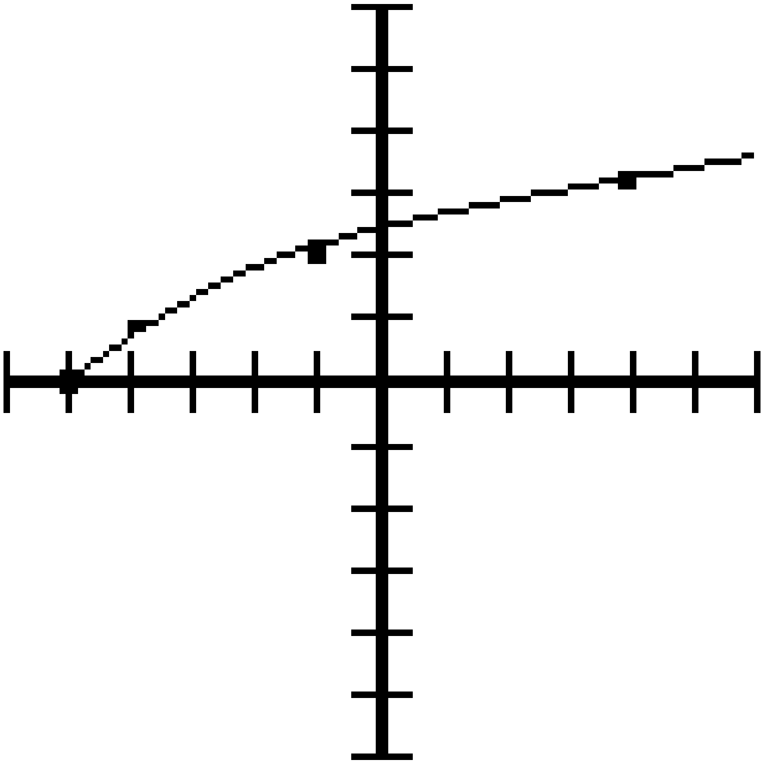
Figure 10.14: (a) $y = f(x)$ (b) $y = f(-\mathrm{x})$; Each point flips around the y-axis

All of these permutations do not need to be memorized: only the general principles need to be understood. But once they are properly understood, even a complex graph such as $y = -2(x+3)^2 + 5$ can be easily graphed. You take the (known) graph of $y = x^2$, flip it over the x-axis (because of the negative sign), stretch it vertically (the 2), move it to the left by 3, and move it up 5.

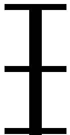
With a good understanding of permutations, and a very simple list of known graphs, it becomes possible to graph a wide variety of important functions. To complete our look at permutations, let's return to the graph of $y = \sqrt{x}$ in a variety of flavors.



(a)



(b)



Chapter 11

Function Concepts – Lines[1]

Most students entering Algebra II are already familiar with the basic mechanics of graphing lines. Recapping very briefly: the equation for a line is $y = \mathrm{mx} + b$ where b is the y-intercept (the place where the line crosses the y-axis) and m is the slope. If a linear equation is given in another form (for instance, $4x + 2y = 5$), the easiest way to graph it is to rewrite it in $y = \mathrm{mx} + b$ form (in this case, $y = -2x + 2\frac{1}{2}$).

There are two purposes of reintroducing this material in Algebra II. The first is to frame the discussion as **linear functions modeling behavior**. The second is to deepen your understanding of the important concept of slope.

Consider the following examples. Sam is a salesman—he earns a commission for each sale. Alice is a technical support representative—she earns $100 each day. The chart below shows their bank accounts over the week.

After this many days (t)	**Sam's bank account (S)**	**Alice's bank account (A)**
0 (*what they started with)	$75	$750
1	$275	$850
2	$375	$950
3	$450	$1,050
4	$480	$1,150
5	$530	$1,250

Table 11.1

Sam has some extremely good days (such as the first day, when he made $200) and some extremely bad days (such as the second day, when he made nothing). Alice makes exactly $100 every day.

Let d be the number of days, S be the number of dollars Sam has made, and A be the number of dollars Alice has made. Both S and A are functions of time. But $s(t)$ is **not a linear function**, and $A(t)$**is a linear function**.

Definition 11.1: Linear Function
A function is said to be "linear" if **every time the independent variable increases by 1, the dependent variable increases or decreases by the same amount**.

Once you know that Alice's bank account function is linear, there are only two things you need to know before you can predict her bank account on any given day.

- How much money she started with ($750 in this example). This is called the y-**intercept**.

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[1] This content is available online at <http://cnx.org/content/m18197/1.3/>.

- How much she makes each day ($100 in this example). This is called the **slope**.

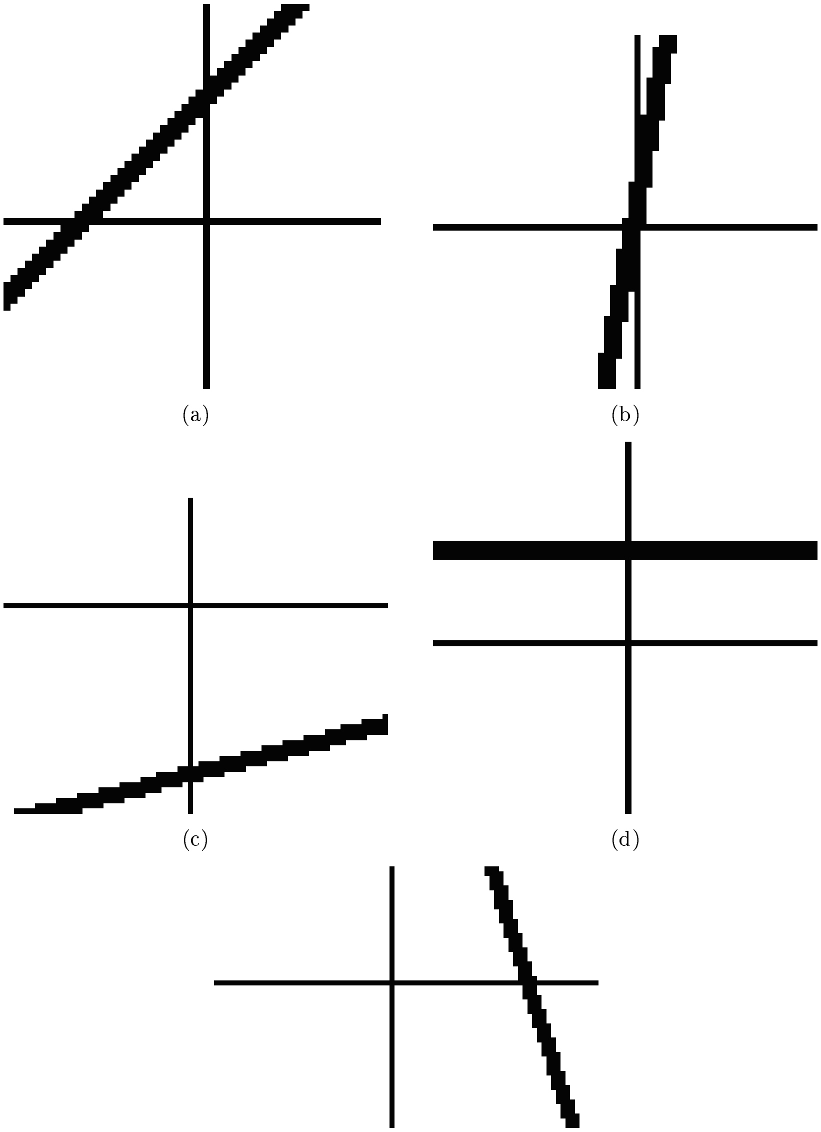
y-intercept is relatively easy to understand. Verbally, it is where the function starts; graphically, it is where the line crosses the y-axis.

But what about slope? One of the best ways to understand the idea of slope is to convince yourself that all of the following definitions of slope are actually the same.

Definitions of Slope		
In our example	**In general**	**On a graph**
Each day, Alice's bank account increases by 100. So the slope is 100.	Each time the independent variable increases by 1, the dependent variable increases by the slope.	Each time you move to the **right** by 1, the graph goes **up** by the slope.
Between days 2 and 5, Alice earns $300 in 3 days. 300/3=100.Between days 1 and 3, she earns $200 in 2 days. 200/2=100.	Take any two points. The change in the dependent variable, **divided** by the change in the independent variable, is the slope.	Take any two points. The change in y divided by the change in x is the slope. This is often written as $\frac{\Delta \mathrm{y}}{\Delta \mathrm{x}}$, or as $\frac{\mathrm{rise}}{\mathrm{run}}$
The higher the slope, the faster Alice is making moey.	The higher the slope, the faster the dependent variable increases.	The higher the slope, the faster the graph rises as you move to the right.

Table 11.2

So slope does not tell you where a graph is, but how quickly it is rising. Looking at a graph, you can get an approximate feeling for its slope without any numbers. Examples are given below.



(a)

(b)

(c)

(d)

(e)

Figure 11.1: (a) A slope of 1: each time you go **over** 1, you also go **up** 1 (b) A steep slope of perhaps 3 or 4 (c) A gentle slope of perhaps $\frac{1}{2}$. (d) A horizontal line has a slope of 0: each time you go **over** 1, you don't go **up** at all! (e) This goes **down** as you move left to right. So the slope is negative. It is steep: maybe a –2.

Chapter 12

Function Concepts – Composite Functions[1]

You are working in the school cafeteria, making peanut butter sandwiches for today's lunch.

- The more classes the school has, the more children there are.
- The more children there are, the more sandwiches you have to make.
- The more sandwiches you have to make, the more pounds (lbs) of peanut butter you will use.
- The more peanut butter you use, the more money you need to budget for peanut butter.

...and so on. Each sentence in this little story is a function. Mathematically, if c is the number of classes and h is the number of children, then the first sentence asserts the existence of a function $h(c)$.

The principal walks up to you at the beginning of the year and says "We're considering expanding the school. If we expand to 70 classes, how much money do we need to budget? What if we expand to 75? How about 80?" For each of these numbers, you have to calculate each number from the previous one, until you find the final budget number.

# Classes →		# Children →		# Sandwiches →		lbs →		$$ →

Table 12.1

But going through this process each time is tedious. What you want is one function that puts the entire chain together: "You tell me the number of classes, and I will tell you the budget."

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[1] This content is available online at <http://cnx.org/content/m18187/1.2/>.

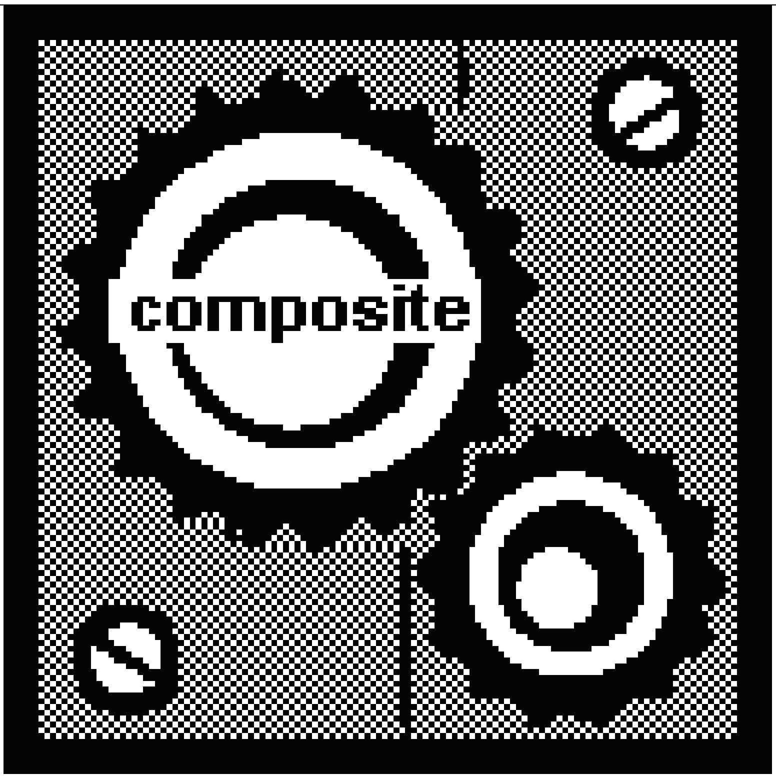
# Classes →		$$ →

Table 12.2

This is a **composite function**—a function that represents in one function, the results of an entire **chain of dependent functions**. Since such chains are very common in real life, finding composite functions is a very important skill.

12.1 How do you make a composite Function?

We can consider how to build composite functions into the function game that we played on the first day. Suppose Susan takes any number you give her, quadruples it, and adds 6. Al takes any number you give him and divides it by 2. Mathematically, we can represent the two functions like this:

$$S(x) = 4x + 6 \tag{12.1}$$

$$A(x) = \frac{x}{2} \tag{12.2}$$

To create a chain like the one above, we give a number to Susan; she acts on it, and gives the resulting number to Al; and he then acts on it and hands back a third number.

3 → Susan → $S(3) = 18$ → Al → $A(18) = 9$

In this example, we are plugging $S(3)$—in other words, 18— into Al's function. In general, for any x that comes in, we are plugging $S(x)$ **into** $A(x)$. So we could represent the entire process as $A(S(x))$. This notation for composite functions is really nothing new: it means that you are plugging $S(x)$ into the A function.

But in this case, recall that $S(x) = 4x + 6$. So we can write:

$$A(S(x)) = \frac{S(x)}{2} = \frac{4x+6}{2} = 2x + 3 \tag{12.3}$$

What happened? We've just discovered a shortcut for the entire process. When you perform the operation $A(S(x))$—that is, when you perform the Al function on the result of the Susan function—you are, in effect, doubling and adding 3. For instance, we saw earlier that when we started with a 3, we ended with a 9. Our composite function does this in one step:

$3 \rightarrow 2x + 3 \rightarrow 9$

Understanding the meaning of composite functions requires real thought. It requires understanding the idea that this variable depends on that variable, which in turn depends on the other variable; and how that idea is translated into mathematics. **Finding** composite functions, on the other hand, is a purely mechanical process—it requires practice, but no creativity. Whenever you are asked for $f(g(x))$, just plug the $g(x)$ function into the $f(x)$ function and then simplify.

Example 12.1: Building and Testing a Composite Function

$f(x) = x^2 - 4x$

$g(x) = x + 2$

What is $f(g(x))$?

- To find the composite, plug $g(x)$ into $f(x)$, just as you would with any number.

$f(g(x)) = (x+2)^2 - 4(x+2)$

- Then simplify.

$f(g(x)) = \left(x^2 + 4x + 4\right) - (4x + 8)$

$f(g(x)) = x^2 - 4$

- Let's test it. $f(g(x))$ means do g, then f. What happens if we start with $x = 9$?

$7 \rightarrow g(x) \rightarrow 7 + 2 = 9 \rightarrow f(x) \rightarrow (9)^2 - 4(9) = 45$

- So, if it worked, our **composite function** should do all of that in one step.

$7 \rightarrow x^2 - 4 = (7)^2 - 4 = 45$ ✓ It worked!

There is a different notation that is sometimes used for composite functions. This book will consistently use $f(g(x))$ which very naturally conveys the idea of "plugging $g(x)$ into $f(x)$." However, you will sometimes see the same thing written as $f \circ g(x)$, which more naturally conveys the idea of "doing one function, and then the other, in sequence." The two notations mean the same thing.

Chapter 13

Function Concepts – Inverse Functions[1]

Let's go back to Alice, who makes \$100/day. We know how to answer questions such as "After 3 days, how much money has she made?" We use the function $m(t) = 100t$.

But suppose I want to ask the reverse question: "If Alice has made \$300, how many hours has she worked?" This is the job of an inverse function. It gives the same relationship, but reverses the dependent and independent variables. $t(m) = m/100$. Given any amount of money, divide it by 100 to find how many days she has worked.

If a function answers the question: "Alice worked this long, how much money has she made?" **then its inverse answers the question:** "Alice made this much money, how long did she work?"

If a function answers the question: "I have this many spoons, how much do they weigh?" **then its inverse answers the question:** "My spoons weigh this much, how many do I have?"

If a function answers the question: "How many hours of music fit on 12 CDs?" **then its inverse answers the question:** "How many CDs do you need for 3 hours of music?"

13.1 How do you recognize an inverse function?

Let's look at the two functions above:

$$m(t) = 100t \tag{13.1}$$

$$t(m) = m/100 \tag{13.2}$$

Mathematically, you can recognize these as inverse functions because they **reverse the inputs and the outputs**.

$3 \rightarrow m(t) = 100t \rightarrow 300$
$300 \rightarrow t(m) = m/100 \rightarrow 3$
✓ **Inverse functions**

Table 13.1

Of course, this makes logical sense. The first line above says that "If Alice works 3 hours, she makes \$300." The second line says "If Alice made \$300, she worked 3 hours." It's the same statement, made in two different ways.

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[1] This content is available online at <http://cnx.org/content/m18198/1.4/>.

But this "reversal" property gives us a way to test any two functions to see if they are inverses. For instance, consider the two functions:

$$f\left(x\right)=3x+7 \tag{13.3}$$

$$g\left(x\right)=\frac{1}{3}x-7 \tag{13.4}$$

They **look** like inverses, don't they? But let's test and find out.

$2\rightarrow 3x+7\rightarrow 13$
$13\rightarrow\frac{3}{x}-7\rightarrow\frac{13}{3}-7\rightarrow-\frac{8}{3}$
× **Not inverse functions**

Table 13.2

The first function turns a 2 into a 13. But the second function does **not** turn 13 into 2. So these are not inverses.

On the other hand, consider:

$$f\left(x\right)=3x+7 \tag{13.5}$$

$$g\left(x\right)=\frac{1}{3}\left(x-7\right) \tag{13.6}$$

Let's run our test of inverses on these two functions.

$2\rightarrow 3x+7\rightarrow 13$
$13\rightarrow\frac{1}{3}\left(x-7\right)\rightarrow 2$
✓ **Inverse functions**

Table 13.3

So we can see that these functions do, in fact, reverse each other: they are inverses.

A common example is the Celsius-to-Fahrenheit conversion:

$$F\left(C\right)=\left(\frac{9}{5}\right)C+32 \tag{13.7}$$

$$C\left(F\right)=\left(\frac{5}{9}\right)\left(F-32\right) \tag{13.8}$$

where C is the Celsius temperature and F the Fahrenheit. If you plug $100\,^{\circ}C$ into the first equation, you find that it is $212\,^{\circ}F$. If you ask the second equation about $212\,^{\circ}F$, it of course converts that back into $100\,^{\circ}C$.

13.2 The notation and definition of an inverse function

The notation for the inverse function of $f(x)$ is $f^{-1}(x)$. This notation can cause considerable confusion, because it **looks like** an exponent, but it isn't. $f^{-1}(x)$ simply means "the inverse function of $f(x)$." It is defined formally by the fact that if you plug any number x into one function, and then plug the result into the other function, you get back where you started. (Take a moment to convince yourself that this is the same definition I gave above more informally.) We can represent this as a composition function by saying that $f\left(f^{-1}(x)\right) = x$.

Definition 13.1: Inverse Function
$f^{-1}(x)$ is defined as the **inverse function** of $f(x)$ if it consistently reverses the $f(x)$ process. That is, if $f(x)$ turns a into b, then $f^{-1}(x)$ must turn b into a. More concisely and formally, $f^{-1}(x)$ is the inverse function of $f(x)$ if $f\left(f^{-1}(x)\right) = x$.

13.3 Finding an inverse function

In examples above, we saw that if $f(x) = 3x + 7$, then $f^{-1}(x) = \frac{1}{3}(x - 7)$. We also saw that the function $\frac{1}{3}x - 7$, which may have looked just as likely, did **not** work as an inverse function. So in general, given a function, how do you find its inverse function?

Remember that an inverse function reverses the inputs and outputs. When we graph functions, we always represent the incoming number as x and the outgoing number as y. So to find the inverse function, **switch the x and y values, and then solve for** y.

Example 13.1: Building and Testing an Inverse Function

1. Find the inverse function of $f(x) = \frac{2x-3}{5}$
   - **a.:** Write the function as $y = \frac{2x-3}{5}$
   - **b.:** Switch the x and y variables. $x = \frac{2y-3}{5}$
   - **c.:** Solve for y. $5x = 2y - 3$. $5x + 3 = 2y$. $\frac{5x+3}{2} = y$. So $f^{-1}(x) = \frac{5x+3}{2}$.
2. Test to make sure this solution fills the definition of an inverse function.
   - **a.:** Pick a number, and plug it into the original function. $9 \rightarrow f(x) \rightarrow 3$.
   - **b.:** See if the inverse function reverses this process. $3 \rightarrow f^{-1}(x) \rightarrow 9$. ✓ It worked!

Were you surprised by the answer? At first glance, it seems that the numbers in the original function (the 2, 3, and 5) have been rearranged almost at random.

But with more thought, the solution becomes very intuitive. The original function $f(x)$ described the following process: **double a number, then subtract 3, then divide by 5**. To reverse this process, we need to reverse each step in order: **multiply by 5, then add 3, then divide by 2**. This is just what the inverse function does.

13.4 Some functions have no inverse function

Some functions have no inverse function. The reason is the rule of consistency.

For instance, consider the function $y = x^2$. This function takes both 3 and –3 and turns them into 9. No problem: a function is allowed to turn different **inputs** into the same **output**. However, what does that say about the inverse of this particular function? In order to fulfill the requirement of an inverse function, it would have to take 9, and turn it into both 3 and –3—which is the one and only thing that functions are **not** allowed to do. Hence, the inverse of this function would not be a function at all!

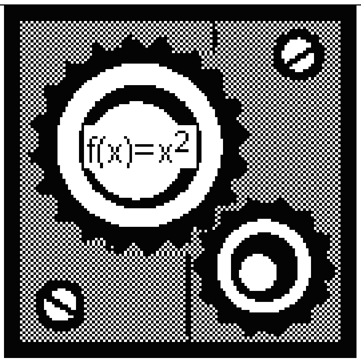
3 →		9 →
−3 →		9 →

Table 13.4: If 3 goes in, 9 comes out. If –3 goes in, 9 also comes out. No problem:

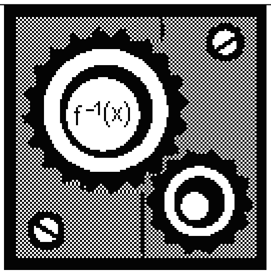
9 →		3 →
9 →		−3 →

Table 13.5: But its inverse would have to turn 9 into both 3 and –3. No function can do this, so there is no inverse.

In general, any function that turns multiple inputs into the same output, does not have an inverse function.

What does that mean in the real world? If we can convert Fahrenheit to Celsius, we must be able to convert Celsius to Fahrenheit. If we can ask "How much money did Alice make in 3 days?" we must surely be able to ask "How long did it take Alice to make $500?" When would you have a function that cannot be inverted?

Let's go back to this example:

Recall the example that was used earlier: "Max threw a ball. The height of the ball depends on how many seconds it has been in the air." The two variables here are h (the height of the ball) and t (the number of seconds it has been in the air). The function $h(t)$ enables us to answer questions such as "After 3 seconds, where is the ball?"

The inverse question would be "At what time was the ball 10 feet in the air?" The problem with that question is, it may well have **two answers**!

The ball is here...	**...after this much time has elapsed**
10 ft	2 seconds (*on the way up)
10 ft	5 seconds (*on the way back down)

Table 13.6

So what does that mean? Does it mean we can't ask that question? Of course not. We can ask that question, and we can expect to mathematically find the answer, or answers—and we will do so in the quadratic chapter. However, it does mean that **time is not a function of height** because such a “function” would not be consistent: one question would produce multiple answers.

Glossary

D **Domain**

The **domain** of a function is all the numbers that it can successfully act on. Put another way, it is all the numbers that can go into the function.

I **Inverse Function**

$f^{-1}(x)$ is defined as the **inverse function** of $f(x)$ if it consistently reverses the $f(x)$ process. That is, if $f(x)$ turns a into b, then $f^{-1}(x)$ must turn b into a. More concisely and formally, $f^{-1}(x)$ is the inverse function of $f(x)$ if $f\left(f^{-1}(x)\right) = x$.

L **Linear Function**

A function is said to be "linear" if **every time the independent variable increases by 1, the dependent variable increases or decreases by the same amount**.

R **Range**

The **range** of a function is all the numbers that it may possibly produce. Put another way, it is all the numbers that can come **out** of the function.

T **The Vertical Line Test**

If you can draw **any** vertical line that touches a graph in two places, then that graph violates the rule of consistency and therefore does not represent any function.

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Functions
This collection is intended as a review of functions for students enrolled in Math 1508 at the University of Texas at El Paso.

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